

THE APPROACH TO CONSISTENCY IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—In his first book on the Analytic Hierarchy Process, T. L. Saaty left open several mathematical questions about the structure of the set of positive reciprocal matrices. In this paper we consider three of these questions: Given an eigenvector and all matrices which give rise to it, can one go from one of them to any other by making small perturbations in the entries? Given two positive column vectors \mathbf{v} and \mathbf{w} is there a perturbation which carries the set of all positive reciprocal matrices with principal right eigenvector \mathbf{v} to the set of positive reciprocal matrices with principal right eigenvector \mathbf{w} ? Does the set of positive reciprocal $n \times n$ matrices whose left and right principal eigenvectors are reciprocals coincide with the set of consistent matrices for $n \geq 4$?

1. INTRODUCTION

A fundamental aspect of the Analytic Hierarchy Process (AHP) is the use of pairwise comparisons to form positive reciprocal matrices whose right principal eigenvectors are used as accurate summaries of the decision maker's judgments. Thus, it is beneficial to understand as much as possible the set of positive reciprocal matrices. In his book [1] on the AHP, T. L. Saaty left open several mathematical questions about the structure of the set of positive reciprocal matrices. In this note, we consider three of these questions, two of which were explicitly asked in the book, and the other of which was almost enunciated as a conjecture. We then apply the answers to the questions to construct an algorithm for revising pairwise comparison judgments toward consistency.

Recall that a *positive reciprocal matrix* is an $n \times n$ matrix A , all of whose entries are positive and such that $a_{ji} = 1/a_{ij}$, $\forall i, j$ (so, in particular, $a_{ii} = 1$, $\forall i$). As is shown in Ref. [1, Theorems 7-4 and 7-14], A has a unique (up to constant multiple) positive eigenvector \mathbf{v} (right principal eigenvector) whose corresponding eigenvalue λ_{\max} is $\geq n$. A positive reciprocal matrix is called *consistent* if there is a positive vector \mathbf{v} such that $a_{ij} = v_i/v_j$, $\forall i, j$. Theorem 7-15 of Ref. [1] states that a positive reciprocal matrix is consistent if and only if $\lambda_{\max} = n$.

We are interested in the set of positive reciprocal matrices taken as a whole and, in particular, the structure of the sets of positive reciprocal matrices which share a common right principal eigenvector. On p. 189 in Ref. [1], Saaty makes the following remark, which contains two fundamental questions about these sets:

“REMARK. Note that we have a many-to-one correspondence between pairwise comparison matrices and eigenvectors. This is fortunate as it allows one to make tradeoffs between attributes and still obtain the same eigenvector for an answer. Therefore, we can obtain the same result from a variety of points of view, and thus choose those matrices which we favor. Otherwise, the universe of experiences would be reduced to a small set of attributes with fixed relative scale values. Relations and their intensity would be deterministic and individual choice would be nonexistent. Of course, this would not introduce conflict. But variety with conflict is richer than determinism. The technical question is: given an eigenvector and all matrices which give rise to it, can one go from one of them to any other by making small perturbations in the entries? In particular, is it possible to go from the matrix of ratios to any other by small perturbations? Another question is: consider two eigenvectors that are small perturbations of each other. Do there exist small perturbations which carry one class of corresponding matrices to the other?”

We will provide affirmative answers to these questions.

The other question we would like to consider involves the relationship between the left and right principal eigenvectors of positive reciprocal matrices. Saaty notes in Theorem 7-33 of Ref. [1] that “the normalized left eigenvector components of a reciprocal positive 3 by 3 matrix are the reciprocals

of the normalized right eigenvector components". On the other hand, "the normalized reciprocal relationship between the left and right eigenvector components no longer holds for $n = 4 \dots$ ". He goes on to say that "one is tempted to conjecture that the reciprocal property between principal left and right eigenvector components holds if and only if the matrix is consistent for $n \geq 4$." This almost-conjecture is false, even for $n = 4$, as illustrated by the following counterexample:

$$\begin{bmatrix} 1 & 1.6 & 0.25 & 4 \\ 0.625 & 1 & 0.625 & 10 \\ 4 & 1.6 & 1 & 4 \\ 0.25 & 0.1 & 0.25 & 1 \end{bmatrix}.$$

This matrix has right principal eigenvector $(4, 5, 8, 1)^T$ and left principal eigenvector $(1/4, 1/5, 1/8, 1)$, but it is easily shown to be not consistent. We give a general principle for constructing counterexamples below.

We review the three questions in the order in which they will be answered in Section 2 below:

1. (Saaty's second question). Given two positive column vectors \mathbf{v} and \mathbf{w} , is there a perturbation which carries the set of all positive reciprocal matrices with principal right eigenvector \mathbf{v} to the set of positive reciprocal matrices with principal right eigenvector \mathbf{w} ?
2. (Saaty's almost-conjecture) Does the set of positive reciprocal $n \times n$ matrices whose left and right principal eigenvectors are reciprocals coincide with the set of consistent matrices for $n \geq 4$?
3. (Saaty's first question, revised) Given an $n \times n$ positive reciprocal matrix with right principal eigenvector \mathbf{v} , is it possible to find a smooth (continuous) path to any other such matrix (in particular, to the unique consistent matrix with principal right eigenvector \mathbf{v}), without ever leaving the set of positive reciprocal matrices with principal right eigenvector \mathbf{v} ?

2. THE GEOMETRY OF THE SET OF POSITIVE RECIPROCAL MATRICES

Our goal in this section is to gain some understanding of the structure of the set \mathcal{P} of positive reciprocal matrices. In particular, we wish to investigate subsets of \mathcal{P} consisting of matrices which all have the same principal right eigenvector. We begin from a group-theoretic point of view. In mathematics, especially geometry, when one considers a certain class of objects that exist as a subset of a larger class, it is natural to ask which natural transformations of the larger class exist that preserve the small class under consideration. This set of transformations comprises a group, which is often called the *structure group* of the subclass. In the AHP, we can consider the class \mathcal{P} of positive reciprocal $n \times n$ matrices as a subset of the set \mathcal{M} of all $n \times n$ matrices. The general linear group $GL(n)$ of invertible $n \times n$ matrices acts on \mathcal{M} in a natural way by *conjugation*: for $A \in GL(n)$ and $M \in \mathcal{M}$, we set

$$I_A(M) = AMA^{-1}.$$

To compute the structure group of \mathcal{P} we need to consider the question: For which invertible matrices A is APA^{-1} a positive reciprocal matrix whenever P is? The answer is given by the following theorem.

Theorem 2.1

The structure group G of the set of positive reciprocal $n \times n$ matrices has $2n!$ connected components. It consists of nonnegative matrices which have exactly one nonzero entry in each row and column, i.e. the matrices can be expressed as $D \cdot S$, where D is a diagonal matrix with positive diagonal entries and S is a permutation matrix, and the negatives of such matrices. The connected component G_0 of the identity consists of diagonal matrices with positive entries on the diagonal.

The proof of this theorem is neither difficult (for the initiated) nor enlightening and so is omitted. The aspect of this which is enlightening is the action of G on \mathcal{P} .

Theorem 2.2

If $P \in \mathcal{P}$ is a positive reciprocal matrix with principal right eigenvector $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ and $D \in G_0$ is a diagonal matrix with positive diagonal entries d_1, d_2, \dots, d_n then $I_D(P) = DPD^{-1}$ is a positive reciprocal matrix with principal eigenvector $\mathbf{w}' = (d_1 w_1, d_2 w_2, \dots, d_n w_n)^T$. The principal eigenvalue is the same for both matrices.

Proof. Compute

$$DPD^{-1}\mathbf{w}' = DP\mathbf{w} = D\lambda\mathbf{w} = \lambda\mathbf{w}'$$

as claimed.

Theorem 2.2 enables us to answer Question 1 of the Introduction in the affirmative. For \mathbf{w} any positive column vector, let $\mathcal{P}_{\mathbf{w}}$ be the set of positive reciprocal matrices with principal right eigenvector \mathbf{w} . Note that $\mathcal{P}_{\mathbf{w}}$ contains exactly one consistent matrix, namely the matrix P with $p_{ij} = w_i/w_j$.

Theorem 2.3

If $\mathbf{v} = (v_1, \dots, v_n)^T$ and $\mathbf{w} = (w_1, \dots, w_n)^T$ are two positive column vectors, then conjugation by the diagonal matrix $D_{\mathbf{v}\mathbf{w}}$ with entries $v_1/w_1, \dots, v_n/w_n$ on the diagonal maps $\mathcal{P}_{\mathbf{w}}$ onto $\mathcal{P}_{\mathbf{v}}$. The corresponding diagonal matrix $D_{\mathbf{w}\mathbf{v}}$ provides the inverse map. Moreover, $D_{\mathbf{v}\mathbf{w}}$ maps the consistent matrix of $\mathcal{P}_{\mathbf{w}}$ to the consistent matrix of $\mathcal{P}_{\mathbf{v}}$.

This is clear from Theorem 2.2. Note that since matrix multiplication is a smooth (differentiable) operation, $\mathcal{P}_{\mathbf{w}}$ is mapped diffeomorphically onto $\mathcal{P}_{\mathbf{v}}$. Also note that conjugation by the one-parameter family of diagonal matrices $D(t)$, where

$$D(t) = \text{diag}(1 - t(1 - v_1/w_1), 1 - t(1 - v_2/w_2), \dots, 1 - t(1 - v_n/w_n))$$

deforms $\mathcal{P}_{\mathbf{w}}$ smoothly to $\mathcal{P}_{\mathbf{v}}$. These provide the perturbations about whose existence Saaty inquired.

We may use Theorems 2.2 and 2.3 to do more than answer Saaty's perturbation question. Since Theorem 2.3 implies that all the sets $\mathcal{P}_{\mathbf{v}}$ have the same structure, we conclude that in order to prove general statements about the structure of all the sets $\mathcal{P}_{\mathbf{v}}$ which do not explicitly involve the value of \mathbf{v} , it is sufficient to prove them for some specific $\mathcal{P}_{\mathbf{v}_0}$. In the proof, it is permissible to use explicitly the value of \mathbf{v}_0 . We give two illustrations of this here, by answering the other two questions from the introduction.

We tackle the question of whether the left and right principal eigenvectors of an inconsistent positive reciprocal $n \times n$ matrix can be reciprocals for $n \geq 4$. First, we note that if a given positive reciprocal matrix P has reciprocal eigenvectors, then so does $I_D(P)$ for any $D \in G_0$. This is because, just as the entries of the right eigenvector are multiplied by the corresponding diagonal entries of D , the elements of the left eigenvector are divided by the corresponding diagonal entries of D . We conclude that to answer the question in general, it is sufficient to consider matrices whose principal right eigenvector is $\mathbf{1} = (1, 1, \dots, 1)^T$. The specific problem at hand becomes to determine whether the principal left eigenvector can be $\mathbf{1}^T = (1, 1, \dots, 1)$ without the matrix being consistent. Since the row eigenvalue equals the column eigenvalue, we see that the eigenvectors are $\mathbf{1}$ and $\mathbf{1}^T$ if and only if all the row sums of the matrix are equal and also equal all of the column sums of the matrix. The question is now whether this implies that all of the entries of the matrix must equal 1.

We begin with $n = 4$. If our matrix is

$$\begin{bmatrix} 1 & a & b & c \\ 1/a & 1 & d & e \\ 1/b & 1/d & 1 & f \\ 1/c & 1/e & 1/f & 1 \end{bmatrix} \quad (1)$$

then we need to check whether the equations

$$\begin{aligned} a + b + c &= \frac{1}{a} + d + e = \frac{1}{b} + \frac{1}{d} + f = \frac{1}{c} + \frac{1}{e} + \frac{1}{f} \\ &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + \frac{1}{d} + \frac{1}{e} = b + d + \frac{1}{f} = c + e + f \end{aligned}$$

imply that $a = b = c = d = e = f = 1$. We can eliminate e and f via

$$e = a + b + c - \frac{1}{a} - d,$$

$$\frac{1}{e} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - a - \frac{1}{d},$$

$$f = a + b + c - \frac{1}{b} - \frac{1}{d}$$

and

$$\frac{1}{f} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - b - d.$$

But then the fact that $a + b + c = \frac{1}{c} + \frac{1}{e} + \frac{1}{f} = c + e + f = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ reduces to the equation

$$c + \frac{1}{c} = d + \frac{1}{d},$$

i.e. $c^2d + d - cd^2 - c = 0 = (d - c)(cd - 1)$. Thus either $c = d$ or $c = \frac{1}{d}$. If $c = d$, then the expressions for e and $\frac{1}{e}$ above yield

$$\left(a + b - \frac{1}{a}\right)\left(\frac{1}{a} + \frac{1}{b} - a\right) = 1, \quad (2)$$

which reduces to

$$(ab - 1)(a - 1)(a + 1)(a + b) = 0. \quad (3)$$

So either $a = \frac{1}{b}$ or $a = 1$ (since a and b must both be positive). If $a = \frac{1}{b}$ we find (by considering the expressions for f and $\frac{1}{f}$) that the matrix must have the form

$$\begin{bmatrix} 1 & a & 1/a & 1 \\ 1/a & 1 & 1 & a \\ a & 1 & 1 & 1/a \\ 1 & 1/a & a & 1 \end{bmatrix}. \quad (4)$$

This is consistent only if $a = 1$, and so yields counterexamples to Saaty's "conjecture" if $a \neq 1$. Considering all the other cases yields the other possibilities

$$\begin{bmatrix} 1 & 1 & b & 1/b \\ 1 & 1 & 1/b & b \\ 1/b & b & 1 & 1 \\ b & 1/b & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & a & 1 & 1/a \\ 1/a & 1 & a & 1 \\ 1 & 1/a & 1 & a \\ a & 1 & 1/a & 1 \end{bmatrix}.$$

These last examples can also be obtained from matrix (4) by conjugation with appropriate permutation matrices. All possible examples of 4×4 matrices with reciprocal eigenvectors are obtained from these by conjugation with elements of G . The example in the introduction was constructed by setting $a = 2$ in matrix (4) and then conjugating by $D = \text{diag}(4, 5, 8, 1)$.

To construct examples of such matrices for larger n is now an easy exercise. We illustrate the idea for $n = 5$. We will form a reciprocal matrix with all 1's except for one entry a and one $\frac{1}{a}$ in each row and column. For instance, we could take

$$\begin{bmatrix} 1 & a & 1 & 1/a & 1 \\ 1/a & 1 & a & 1 & 1 \\ 1 & 1/a & 1 & 1 & a \\ a & 1 & 1 & 1 & 1/a \\ 1 & 1 & 1/a & a & 1 \end{bmatrix}. \quad (5)$$

With $n = 5$ it is possible to use $a, \frac{1}{a}, b$ and $\frac{1}{b}$ as follows:

$$\begin{bmatrix} 1 & a & b & 1/a & 1/b \\ 1/a & 1 & a & 1/b & b \\ 1/b & 1/a & 1 & b & a \\ a & b & 1/b & 1 & 1/a \\ b & 1/b & 1/a & a & 1 \end{bmatrix}. \quad (6)$$

It turns out that this matrix and its conjugates by elements of G are the only examples of matrices with reciprocal eigenvectors for $n = 5$. We leave it to the reader to construct larger counterexamples.

We now turn to the last question, namely whether the sets \mathcal{P}_v are connected. We will, in fact, show something much more precise.

Theorem 2.4

For each positive vector v , the set \mathcal{P}_v of positive reciprocal $n \times n$ matrices with right principal eigenvector v is diffeomorphic to $\mathbf{R}^{(n-1)(n-2)/2}$.

To begin, we again reduce to the case where $v = \mathbf{1} = (1, 1, \dots, 1)^T$, since \mathcal{P}_1 is diffeomorphic to \mathcal{P}_w (the diffeomorphism is conjugation by D_{1w}) for each positive w . Thus, the result will follow in all its generality if we can prove it for \mathcal{P}_1 . Recall that \mathcal{P}_1 is the set of positive reciprocal matrices whose row sums are all equal (to the principal eigenvalue). There is a unique consistent matrix in \mathcal{P}_1 , namely the matrix whose entries are all equal to 1. The theorem will follow easily from the following.

Lemma 2.1

Each choice of positive numbers $\{a_{ij} > 0 \mid 2 \leq i < j \leq n\}$ uniquely determines an element of \mathcal{P}_1 .

Proof. Given such a set of $\frac{1}{2}(n-1)(n-2)$ numbers, we form a positive reciprocal $n \times n$ matrix

P with $\mathbf{1}$ as its right eigenvector as follows. Set $p_{ij} = a_{ij}$ for $2 \leq i < j \leq n$ (of course), set $p_{ii} = 1$ for $i = 1, \dots, n$, and set $p_{ji} = 1/a_{ij}$ for $2 \leq i < j \leq n$. To determine $p_{1j} = 1/p_{j1}$, for $j = 2, \dots, n$, we need to solve the system of equations

$$\begin{aligned} \frac{1}{p_{21}} + \frac{1}{p_{31}} + \cdots + \frac{1}{p_{n1}} &= p_{21} + (p_{23} + \cdots + p_{2n}) \\ &= p_{31} + (p_{32} + p_{34} + \cdots + p_{3n}) \\ &\vdots \\ &= p_{n1} + (p_{n2} + \cdots + p_{n,n-1}). \end{aligned} \quad (7)$$

This is a system of $n - 1$ equations for the $n - 1$ unknowns p_{21}, \dots, p_{n1} . For $j = 2, \dots, n$, set $c_j = \left(\sum_{k=2}^n p_{jk} \right) - 1$. We have already determined the c_j , and so equations (7) become

$$\begin{aligned} \frac{1}{p_{21}} + \frac{1}{p_{31}} + \cdots + \frac{1}{p_{n1}} &= p_{21} + c_2 \\ &= p_{31} + c_3 \\ &\vdots \\ &= p_{n1} + c_n. \end{aligned}$$

So we see that $p_{31} = p_{21} + c_2 - c_3, \dots, p_{n1} = p_{21} + c_2 - c_n$. We are reduced to solving the equation

$$\frac{1}{p_{21}} + \frac{1}{p_{21} + c_2 - c_3} + \cdots + \frac{1}{p_{21} + c_2 - c_n} = p_{21} + c_2 \quad (8)$$

for p_{21} , such that

$$\max \{c_j - c_2\}_{j=2}^n < p_{21} < \infty.$$

But note three simple facts:

1. The value of the l.h.s. of equation (8) decreases monotonically as p_{21} increases, while the value of the r.h.s. increases monotonically as p_{21} increases.
2. As $p_{21} \rightarrow \infty$, the l.h.s. of equation (8) approaches zero while the r.h.s. becomes infinite.
3. As $p_{21} \rightarrow \max \{c_j - c_2\}_{j=2}^n$, the l.h.s. of equation (8) approaches $+\infty$ while the r.h.s. remains bounded.

It follows from the intermediate-value theorem that there is a solution of equation (8), and from the mean-value theorem that is unique. Thus the first column of the matrix P has been uniquely determined, and the first row is determined uniquely from that. The lemma is proved.

The proof of the theorem is now obvious: the set \mathcal{P}_1 is completely parametrized by the entries of its matrices above the diagonal in rows 2 through n . The numbers $\log a_{ij}$, $2 \leq i < j \leq n$, are in one-to-one correspondence with $\mathbf{R}^{(n-1)(n-2)/2}$.

3. AN INTERACTIVE APPROACH TO CONSISTENCY

In the AHP, the usual procedure is for the judges to accumulate the results of their pairwise comparisons in a positive reciprocal matrix, and then to accept the resulting eigenvector as a summary of their judgments. Our better understanding of the set of positive reciprocal matrices allows us to use Theorems 2.2 and 2.4 to guide the judges in revising the pairwise comparison matrix toward consistency.

In our approach (as in the usual approach), the judges provide an initial matrix P_0 of pairwise comparisons for which is computed the right eigenvector v_0 . The judges are then given the opportunity to adjust v_0 , which yields a new vector \bar{v}_0 . Theorem 2.2 is then used as follows. We form the diagonal matrix $D_{v_0 v_0}$, and P_0 is conjugated to form a new positive reciprocal matrix \bar{P}_0 . This new matrix is an alternative to the original pairwise comparison matrix, but before it is presented to the judges, Theorem 2.4 is used (as explained below) to make \bar{P}_0 "10% more consistent", and the resulting P_1 is presented to the judges as an alternative to their original P_0 . If this procedure is repeated indefinitely, the limit of the sequence of matrices P_0, P_1, \dots will be a consistent matrix.

We must explain what is meant by "10% more consistent". One can envision many ways of making the matrix more consistent (by reducing the principal eigenvalue by a fixed amount etc.). We will choose a computationally simple approach suggested by the proof of Theorem 2.4. The matrices with a fixed eigenvector v are in one-to-one correspondence with \mathbf{R}^N , for $N = (n-1)(n-2)/2$ as follows. Given a (positive reciprocal) matrix P with eigenvector v , we form D_{1v} , and conjugate to get a matrix P' with eigenvector 1 . The logarithms of the entries of P' above the diagonal in rows 2 through $n-1$ are the N "coordinates" of P . Note that in this system the unique consistent matrix with eigenvector v is at the origin. The definition of "10% more consistent" we will use is to move the coordinates of P one-tenth of the way to zero. In other words, we raise the elements above the diagonal of P' in rows 2 through $n-1$ to the power 0.9. This brings their logarithms 10% closer to zero. We then solve equations (7) for the new first row and column and the resulting matrix \bar{P}' will be closer to the consistent matrix with eigenvector 1 . Finally, we conjugate by D_{v1} to get our new matrix \bar{P} with eigenvector v that is closer to consistency than the original P .

An illustration. In Saaty's "distances to cities from Philadelphia" example [1, p. 42], the original matrix is

$$\begin{bmatrix} 1 & 1/3 & 8 & 3 & 3 & 7 \\ 3 & 1 & 9 & 3 & 3 & 9 \\ 1/8 & 1/9 & 1 & 1/6 & 1/5 & 2 \\ 1/3 & 1/3 & 6 & 1 & 1/3 & 6 \\ 1/3 & 1/3 & 5 & 3 & 1 & 6 \\ 1/7 & 1/9 & 1/2 & 1/6 & 1/6 & 1 \end{bmatrix}.$$

Thus, the initial eigenvector is $v = (0.263, 0.397, 0.033, 0.116, 0.164, 0.027)^T$, with eigenvalue 6.45. Now, suppose that the person making the judgments knows some exact information about the distances from Philadelphia to, say, Chicago and Montreal (v_3 and v_6 , respectively). He might know that their ratio should be 33/20. He could then ask that the entry v_6 be decreased to 0.020. We would then want to produce a positive reciprocal matrix with eigenvector $v' = (0.263, 0.397, 0.033, 0.116, 0.164, 0.020)^T$. To do this, we conjugate the original matrix with D_{1v} . We then raise the elements of the matrix so obtained to the power 0.9 (except for the first row and column), and then complete the first row and column to preserve the eigenvector 1 . Then, we take this matrix and conjugate with $D_{v'1}$, to obtain the matrix

$$\begin{bmatrix} 1.000 & 0.357 & 8.147 & 2.923 & 2.771 & 9.774 \\ 2.803 & 1.000 & 9.254 & 3.039 & 2.936 & 12.655 \\ 0.123 & 0.108 & 1.000 & 0.176 & 0.200 & 2.552 \\ 0.342 & 0.329 & 5.682 & 1.000 & 0.359 & 7.770 \\ 0.361 & 0.341 & 4.991 & 2.782 & 1.000 & 8.043 \\ 0.102 & 0.079 & 0.392 & 0.129 & 0.124 & 1.000 \end{bmatrix},$$

which has eigenvalue 6.36 and eigenvector v' . To continue the process, the judge might then consider the new matrix of pairwise comparisons, to see if any of the entries are at odds with his actual opinion. This process of revision of the eigenvector as well as the comparison matrix might serve

to lead a group gently toward an acceptably consistent comparison matrix; it also serves to give the judge an opportunity to make finer comparisons than are allowed by the verbal 1–9 comparison scale which is often used as a starting point for constructing comparison matrices.

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REFERENCE

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